CHAPTER 11: Infinite Series

§11.3 The integral Test

Section Contents:

(1) Series with non-negative terms.

(2) The integral test.

1. Series with non-negative terms.

Definition 1. A series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_k + \cdots,$$

where each $a_k \geq 0$ is called a **series with non-negative terms**.

A series with non-negative terms $\sum_{k=1}^{\infty} a_k$ has a non-decreasing sequence of partial sums $\{S_n\}$:

$$S_1 \leq S_2 \leq \cdots \leq S_n \leq \cdots.$$

Thus the series converges if and only if the sequence $\{S_n\}$ is bounded above, i.e.

$$S_n \leq M \qquad (n \geq 1)$$

for some value M.

1. The integral test. Suppose

(a) f is continuous real function on an interval $[N, \infty)$, where $N > 0$.

(b) $f(x) \geq 0$ for all $x \geq N$.

(c) f is non-increasing on $[N, \infty)$.

(d) $a_n = f(n)$ for all $n \geq N$.

Then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Leftrightarrow \int_N^{\infty} f(x)\, dx \text{ converges.}$$

Sketch of proof. Textbook: Figure 8.13.

2. Harmonic and $p-$series

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{k^p} + \cdots,$$

where p is a fixed real, is called a $p-$series. If $p = 1$, the series becomes

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} + \cdots$$

which is known as the **harmonic series**.

Theorem 1. (**The $p-$series test**)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges} & \text{for } p > 1 \\ \text{diverges} & \text{for } p \leq 1 \end{cases}$$

Example 1. The $p-$series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

converge since $2, 3/2 > 1$, and the series

$$\sum_{k=1}^{\infty} \frac{1}{k}, \quad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^{e/\pi}}$$

diverge since $1, 1/2, e/\pi \leq 1$.

Proof of Theorem 1. We consider two cases:

1. $p \leq 0$. In this case,

$$\lim_{k \to \infty} \frac{1}{k^p} = \lim_{k \to \infty} k^{-p} = \begin{cases} 1 & \text{if } p = 0 \\ \infty & \text{for } p < 0 \end{cases}$$

and the $p-$series diverges.

2. $p > 0$. In this case the function

$$f(x) = \frac{1}{x^p} \qquad (x \geq 1),$$

satisfies the conditions $(a), \cdots, (d)$ of the integral test.

Consider the improper integral

$$\int_1^\infty \frac{1}{x^p}\, dx.$$

We consider three cases:

(a) $p = 1$. In this case the integral becomes

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\, dx = \lim_{t\to\infty} \ln t = \infty,$$

and the harmonic series diverges.

(b) $0 < p < 1$. In this case the integral becomes

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p} dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{1}{1-p} \lim_{t\to\infty} \left(t^{1-p} - 1\right) = \infty,$$

and the $p-$series (harmonic series) diverges.

(c) $p > 1$. In this case and in view of (b) we have:

$$\int_1^\infty \frac{1}{x^p}\, dx \;=\; \frac{1}{p-1} \lim_{t\to\infty} \left(1 - \frac{1}{t^{p-1}}\right) = \frac{1}{p-1},$$

and the $p-$series converges.