CHAPTER 11: Infinite Series

§11.4 Comparison Tests

Theorem 1. (The Direct comparison test) Suppose

$$0 \leq a_k \leq b_k \qquad (k \geq 1).$$

(a) $\sum b_k$ converges $\Rightarrow \sum a_k$ converges.

(b) $\sum a_k$ diverges $\Rightarrow \sum b_k$ diverges.

Sketch of Proof. Let

$S_n = a_1 + a_2 + \cdots + a_n$ and $T_n = b_1 + b_2 + \cdots + b_n$.

(a) $\sum_1^\infty b_k$ converges $\Rightarrow S_n \leq T_n \leq M$ for some M. This proves (a).

(b) $\sum_1^\infty a_k$ diverges $\Rightarrow T_n \geq S_n \to \infty$ as $n \to \infty$. This proves (b).

Example 1. (i) The series $\sum_1^\infty 1/(2k^3+1)$ converges since

$$\frac{1}{2k^3+1} \leq \frac{1}{k^3} \qquad (k \geq 1)$$

and since the $p-$series $\sum_1^\infty 1/k^3$ converges.

(ii) The series $\sum_1^\infty 1/(3k-2)$ diverges since

$$\frac{1}{3k-2} > \frac{1}{3k} \qquad (k \geq 1)$$

and since the harmonic series diverges.

(iii) The series

$$\sum_{1}^{\infty} \frac{k^3}{k^5 + 5k^4 + 7}$$

converges since

$$\frac{k^3}{k^5 + 5k^4 + 7} < \frac{k^3}{k^5} = \frac{1}{k^2}$$

and since the $p-$series $\sum_1^\infty 1/k^2$ converges.

(iv) The series $\sum_1^\infty 1/\ln(k+1)$ diverges since

$$\frac{1}{\ln(k+1)} > \frac{1}{k} \qquad (k \geq 3)$$

and since the harmonic series diverges.

Theorem 2. (The limit comparison test)

Suppose the series

$$\sum a_k \text{ and } \sum b_k \text{ where } a_k \geq 0 \text{ and } b_k > 0.$$

Then

(a) $\lim a_k/b_k = L > 0 \Rightarrow$ both series converge or both series diverge.

(b) $\lim a_k/b_k = 0$ and $\sum b_k$ converges $\Rightarrow \sum a_k$ converges.

(c) $\lim a_k/b_k = \infty$ and $\sum a_k$ converges $\Rightarrow \sum b_k$ converges.

(d) $\lim a_k/b_k = \infty$ and $\sum b_k$ diverges $\Rightarrow \sum a_k$ diverges.

Example 2. (i) The series

$$\sum_{1}^{\infty} \frac{3n^3 - 2n^2 + n}{n^4 + 1}$$

diverges since

$$\lim \frac{(3n^3 - 2n^2 + n)/(n^4 + 1)}{1/n} = 3 > 0$$

and the harmonic series diverges.

(ii) The series $\sum_1^\infty \sin(\pi/k)$ diverges since

$$\lim \sin(\pi/k)/(1/k) = \pi \lim \sin(\pi/k)/(\pi/k) = \pi > 0$$

and the harmonic series diverges.

(iii) The series

$$\sum_{1}^{\infty} \frac{5\sqrt{k}+100}{2k^3\sqrt{k}+9\sqrt{k}}$$

converges since

$$\frac{(5\sqrt{k}+100)/(2k^3\sqrt{k}+9\sqrt{k})}{1/k^3} = \frac{5}{2} > 0$$

and the $p-$series $\sum_1^\infty 1/k^3$ converges.

(iv) The series

$$\sum_{1}^{\infty} \frac{\ln(1+1/k)}{k}$$

converges since

$$\lim \frac{(\ln(1+1/k))/k}{1/k^2} = \lim \frac{\ln(1+1/k)}{1/k}$$

$$= \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{[\ln(1+x)]'}{(x)'} = 1$$

and the $p-$series $\sum_1^\infty 1/k^2$ converges.