CHAPTER 11: Infinite Sequences and Series

§11.1 Sequences

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1. Definition and notation of sequences.

Definition 1. An **infinite sequence of real numbers**, or simply a **sequence**, abbreviated as **seq.**, is a real-valued function whose domain is the set of integers greater than or equal to some positive integer m.

Example 1. The functions

$$a(n) = n^2 \quad n = 0, 1, \cdots;$$

$$b(n) = (-1)^{n+1}/n \quad n = 1, 2, \cdots;$$

$$c(n) = \sqrt{\ln n} \quad n = 1, 2, \cdots;$$

$$d(n) = e^n/n^5 \quad n = 1, 2, \cdots;$$

are sequences of real numbers.

Let **a** be a sequence defined for positive integers $n \geq m$ for some positive integer m. Then the range of the sequence is

$$\{a(m), a(m+1), a(m+2), \cdots\},$$

or

$$\{a(n)\}_{n=m}^{\infty},$$

or simply

$$\{a(n)\}$$

without reference to the upper and lower indices. Usually, the range of **a** as an ordered set is called the sequence **a**.

Traditionally, $a(n)$ is denoted by a_n in which case the sequence $\mathbf{a}$ becomes

$$\{a_m, a_{m+1}, a_{m+2}, \cdots\},$$

or

$$\{a_n\}_{n=m}^{\infty},$$

or simply

$$\{a_n\}.$$

We call $a_m, a_{m+1}, \cdots, a_n, \cdots$ the **terms** of the sequence $\mathbf{a}$; in particular, we call a_n the $n-$**th term**.

Example 2. The sequence whose $n-$th term is $a_n = 1/n$ can be written as

$$\{1, 1/2, 1/3, \cdots, 1/n, \cdots\},$$

or

$$\{1/n\}_{n=1}^{\infty}, \text{ or } \{1/n\}.$$

Also, the sequence whose $n-$th term is $b_n = \sqrt{\ln n}$, where $n \geq 1$, can be written as

$$\{0, \sqrt{\ln 2}, \sqrt{\ln 3}, \cdots, \sqrt{\ln n}, \cdots\},$$

or $\{\sqrt{\ln n}\}_{n=1}^{\infty}$, or $\{\sqrt{\ln n}\}$.

Graphical Representation of sequences.

A sequence $\mathbf{a} = \{\mathbf{a_n}\}_{\mathbf{n=m}}^{\infty}$ is represented graphically in two ways as follows.

(i) By its graph as the function $\mathbf{a}(n) = a_n$, where $n = m, m+1, \cdots$.

(ii) By the graphs of the terms a_n on the real axis.

Example 3. The graphs of the sequence $\{\sqrt{n}\}$ are

2. Convergence and Divergence of Sequences.

Definition 2. A sequence $\{a_n\}$ is said to **converge** to a real number L if for every positive value ϵ there exists a positive integer N such that

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

In this case we write:

$$\lim_{n\to\infty} a_n = L \text{ or } \lim a_n = L$$

or

$$a_n \to L \text{ as } n \to \infty.$$

Graphically this means that in any open interval about L there lies all but finitely many of the terms a_n of the sequence.

the sequence $\{a_n\}$ is said to **diverge** if it does not converge.

Example 4. We show that $\lim c = c$, c is a constant. Fix $\epsilon > 0$, and let N be any positive integer. Since

$$n > N \Rightarrow |c - c| = 0 < \epsilon,$$

$\lim c = c$.

Example 5. We show that $\lim 1/n = 0$. Fix $\epsilon > 0$, and choose a positive integer $N > 1/\epsilon$. Then

$$n > N \Rightarrow \frac{1}{n} < \frac{1}{N} < \epsilon,$$

or

$$n > N \Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon$$

which proves $\lim 1/n = 0$.

Example 6. We show that $\lim a_n = 1/3$, where $a_n = 0.33\cdots 3$ with 3 repeated n times. First observe that for $\epsilon > 0$,

$$\begin{aligned} |a_n - \frac{1}{3}| &= \left|\frac{0.99\cdots 9 - 1}{3}\right| \text{ (9 repeated n times)} \\ &= \frac{1}{3}10^{-n} < \frac{1}{n} < \epsilon \end{aligned}$$

implies $n > 1/\epsilon$. Thus if we choose a positive integer N such that $N > 1/\epsilon$, then

$$\begin{aligned} n > N \quad \Rightarrow \quad & |a_n - \frac{1}{3}| \\ &= |\frac{0.99\cdots 9 - 1}{3}| \text{ (9 repeated n times)} \\ &= \frac{1}{3}10^{-n} \\ &< \frac{1}{n} < \frac{1}{N} < \epsilon. \end{aligned}$$

Therefore, $\lim a_n = 1/3$.

3. Calculating limits of sequences.

Theorem 1. (Algebra of Limits)

If $\lim a_n = A$ and $\lim b_n = B$. Then

- Sum Law: $\lim(a_n \pm b_n) = A \pm B$.
- Product Law: $\lim(a_n b_n) = AB$.
- Quotient Law: $\lim a_n/b_n = A/B$ provided that $B, b_n \neq 0$.
- Multiple Law: $\lim(ka_n) = kA$; k constant.

Example 7. (i)

$$\begin{aligned}\lim \frac{n \pm 5}{n} &= \lim(1 \pm \frac{5}{n}) \\ &= \lim 1 \pm 5 \lim \frac{1}{n} \\ &= 1 \pm (5)(0) = 1.\end{aligned}$$

(ii)

$$\lim \frac{10}{n^2} = 10 \lim \frac{1}{n^2} = 10 \lim \left(\frac{1}{n}\right)^2 = (10)(0) = 0.$$

(iii)

$$\lim \frac{3n^5 - n^2 + 1}{2n^5 - 7} = \lim \frac{3 - 1/n^3 + 1/n^5}{2 - 7/n^5} = \frac{3}{2}.$$

(iv) If a_r and b_s are nonzero, then

$$\lim \frac{a_r n^r + a_{r-1} x^{r-1} + \cdots + a_0}{b_s n^s + a_{s-1} n^{s-1} + \cdots + b_0}$$

equals a_r/b_s if $r = s$, zero if $r < s$, and ∞ if $r > s$ and $a_r/b_s > 0$ and $-\infty$ if $r > s$ and $a_r/b_s < 0$.

Theorem 2. (Sandwich Theorem)

Suppose that

(i) $a_n \leq b_n \leq c_n$ for $n > m$, where m is a fixed positive integer, and if

(ii) $\lim a_n = \lim c_n = L$ for some real number L.

Then $\lim b_n = L$.

Example 8. (i)

$$\lim \frac{\cos n}{n} = 0$$

since

$$\left|\frac{\cos n}{n}\right| \leq \frac{1}{n}.$$

(ii)

$$\lim \frac{(-1)^n}{n} = 0$$

since

$$\left|\frac{(-1)^n}{n}\right| \leq \frac{1}{n}.$$

(iii)

$$\lim \sqrt{4 + \frac{1}{n^2}} = 2$$

since

$$\begin{aligned} 2 = \sqrt{4} \quad &< \quad \sqrt{4 + \frac{1}{n^2}} < \sqrt{4 + \frac{2}{n} + \frac{1}{n^2}} \\ &= \quad \sqrt{(2 + \frac{1}{n})^2} = 2 + \frac{1}{n} \end{aligned}$$

and $\lim 2 = \lim(2 + 1/n) = 2$.

Theorem 3. (Continuous Function Theorem)

Suppose that

(i) $\lim a_n = L$ and

(ii) f is a continuous function at L.

Then $\lim f(a_n) = f(\lim a_n) = f(L)$.

Example 9. (i) $\lim \cos(\pi/n) = \cos(\lim \pi/n) = \cos 0 = 1$.

(ii)

$$\begin{aligned} \lim \sqrt{4 + \frac{1}{n^2}} &= \sqrt{\lim(4 + \frac{1}{n^2})} \\ &= \sqrt{4 + \lim \frac{1}{n^2}} = 2. \end{aligned}$$

(iii) $\lim 3^{1/n} = 3^{\lim 1/n} = 3^0 = 1$.

(iv) $\lim \ln(n/(n+1)) = \ln[\lim(n/(n+1))] = \ln 1 = 0$.

Theorem 4. (L'H$\hat{o}$pital's rule)

Let a be a real number or $\pm\infty$. Suppose

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0, \infty, -\infty,$ respectively, or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \mp\infty.$$

(ii) f and g are differentiable around a.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the right limit exists or is infinite.

Corollary 1. Under the assumptions of Theorem 4, if

$$a_n = \frac{f(n)}{g(n)}, \qquad (n > m)$$

for some positive integer m, then

$$\lim a_n = \lim \frac{f(n)}{g(n)} = \lim \frac{f'(n)}{g'(n)}.$$

Example 9. (i)

$$\begin{aligned}\lim \frac{\ln n}{n} &= \lim_{x\to\infty} \frac{\ln x}{x} \\ &= \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} \\ &= \lim_{x\to\infty} \frac{1/x}{1} = 0.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0. \tag{1}$$

(ii)

$$\begin{aligned}\lim n^{1/n} &= \lim e^{(\ln n)/n} \\ &= e^{\lim(\ln n)/n} = e^0 = 1.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} n^{1/n} = 1. \tag{2}$$

(iii) For a fixed real number x we have:

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty} e^{n\ln(1+x/n)} \\
&= \lim_{n\to\infty} e^{\frac{\ln(1+x/n)}{1/n}} \\
&= \lim_{t\to\infty} e^{\frac{\ln(1+x/t)}{1/t}} \\
&= e^{\lim_{t\to\infty}\frac{[1/(1+x/t)][-x/t^2]}{-1/t^2}} \\
&= e^{\lim_{t\to\infty} x/(1+x/t)} = e^x.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x. \tag{3}$$

If $x = 1$, then (3) gives:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \qquad (-\infty < x < \infty).$$

The following limits can also be shown for a fixed real value x.

$$\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1). \tag{4}$$

$$\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0). \tag{5}$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (-\infty < x < \infty). \tag{6}$$

5. Familiar limits. The following limits are useful for problem solving:

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

$$\lim_{n\to\infty} n^{1/n} = 1.$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1).$$

$$\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0).$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (-\infty < x < \infty).$$

6. **Recursively defined sequences**.

Definition 3. A sequence $\{a_n\}_m^\infty$ is said to be defined **recursively** if its definition consists of

(a) values of one or more terms, and

(b) a **recursive formula** that yields the remaining terms.

Example 10. (i) The recursive definition of the sequence in Example 4(ii) is given by the following:

(a) $a_1 = \sqrt{2}$;

(b) $a_{n+1} = \sqrt{a_n}$ for all $n \geq 1$.

(ii) The sequence $\{a_n\}_1^\infty$ defined recursively by setting

(a) $a_1 = 1$ and

(b) $a_{n+1} = a_n + (-2)^n$ for all $n \geq 1$

has initial terms: $1, -1, 3, -5, 11, \cdots$. Note that the sequence does not converge; for if $\lim a_n \to L$, then, by using the recurrence formula (b),

$$L = L + \lim(-2)^n$$

which gives the false conclusion $\lim(-2)^n = 0$.

(iii) The **Fibonacci** sequence is defined recursively by setting

(a) $a_1 = a_2 = 1$ and

(b) $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 2$.

This sequence has the initial terms

$$1, 1, 2, 3, 5, 8, 13, \cdots$$

Can you find an explicit formula for a_n?

7. Monotone and bounded sequences.

Definition 4. A sequence $\{a_n\}_{n=m}^{\infty}$ is said to be a

(a) **nondecreasing** sequence if

$$a_m \leq a_{m+1} \leq \cdots \leq a_n \leq \cdots$$

or

$$a_n \leq a_{n+1} \qquad (n \geq m).$$

(b) **nonincreasing** sequence if

$$a_m \geq a_{m+1} \geq \cdots \geq a_n \geq \cdots$$

or

$$a_n \geq a_{n+1} \qquad (n \geq m).$$

(c) **monotone** sequence if it is either a nondecreasing or a nonincreasing sequence.

If in (a) and (b) the inequalities were strict, then we may say the sequence $\{a_n\}_{n=m}^{\infty}$ is **strictly increasing**, **strictly decreasing**. In this case we may also say that the sequence is **strictly monotone**.

Example 11. (i) The sequence

$$0.1, 0.11, \cdots, 0.11\cdots 1(1 \text{ n times}), \cdots$$

is nondecreasing (strictly increasing)

(ii) the sequence $\{(\ln n)/n\}_{n=3}^{\infty}$ is nonincreasing (strictly decreasing) since

$$\frac{d}{dx}\frac{\ln x}{x} = \frac{1-\ln x}{x^2} < 0 \qquad (x \geq 3).$$

(iii) the sequence $\{\lfloor -n/2 \rfloor\}_{n=1}^{\infty}$ is nonincreasing since it is the sequence

$$-1, -1, -2, -2, -3, -3, \cdots.$$

Definition 5. A sequence $\{a_n\}_{n=m}^{\infty}$ is said to be

(a) **bounded from above** if there exists a real-value K such that

$$a_n \leq K \qquad\qquad (n \geq m).$$

(b) **bounded from below** if there exists a real-value k such that

$$a_n \geq k \qquad\qquad (n \geq m).$$

(c) **bounded** if it is bounded from above and below; i.e. there exists a real-value M such that

$$|a_n| \leq M \qquad\qquad (n \geq m).$$

Moreover, the sequence $\{a_n\}_m^{\infty}$ is said to have an **upper bound** K if (a) holds, a **lower bound** K if (b) holds, and a **bound** M if (c) holds.

Example 12. (i) The sequence $\{-n^2\}_{n=1}^{\infty}$ is bounded above by zero but1 is not bounded below.

(ii) The sequence $\{\lfloor n/3 \rfloor\}_{n=1}^{\infty}$ is bounded below by zero but is not bounded above.

(iii) The sequence $\{(-1)^n(1-1/n)\}_{n=1}^{\infty}$ is bounded by 1 since

$$|(-1)^n(1-1/n)| = 1 - 1/n \leq 1.$$

Theorem 5. (Monotone Sequence Theorem)

Every bounded monotone sequence is convergent. In particular, a sequence $\{a_n\}$ that is

(a) nondecreasing and bounded above is convergent, and

(b) nonincreasing and bounded below is convergent.

Example 13. (i) The sequence $\{\tan^{-1} n\}_{n=1}^{\infty}$ is bounded above and nondecreasing. Hence it is convergent; in fact, $\lim \tan^{-1} n = \pi/2$.

(ii) The sequence

$$\{2^{1/(2^n)}\}_{n=1}^{\infty} : \sqrt{2}, \sqrt{\sqrt{2}}, \sqrt{\sqrt{\sqrt{2}}}, \cdots$$

is bounded below and nonincreasing. Hence it is convergent. The conclusion can also be obtained by observing that the sequence is a subsequence of $\{2^{1/n}\}$ which converges to 1.