CHAPTER 11: Infinite Series

§11.2 Infinite Series

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1. Series and partial sums.

While it is possible to add two, three, hundred, or even million real numbers, how can we attach a meaning to a sum of infinitely many numbers?

The theory of infinite series is an attempt to answer this question.

To form an infinite series, we begin with a sequence

$$\{a(n)\} : a_1, a_2, \cdots, a_n, \cdots$$

and establish the finite sums

$$
\begin{aligned}
S_1 &= a_1 = \sum_{k=1}^{1} a_k \\
S_2 &= a_1 + a_2 = \sum_{k=1}^{2} a_k \\
&\vdots \\
S_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\
&\vdots
\end{aligned}
$$

This leads us to consider the "infinite sum"

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots$$

which we shall call **infinite series**.

Definition 1. An **infinite series of real numbers**, or simply an **infinite series**, is an expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots.$$

The number a_k is called the $k-$**th term** of the series. The $n-$**th partial sum** of the series is the finite sum

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

We say the series $\sum_{k=1}^{\infty} a_k$ **converges** if the sequence $\{S_n\}$ of partial sums converges. If

$$\lim_{n\to\infty} S_n = S,$$

then we say that the series $\sum_{k=1}^{\infty} a_k$ **converges to** S, or is **convergent to** S, and we write

$$S = \sum_{k=1}^{\infty} a_k.$$

On the other hand, if the sequence $\{S_n\}$ diverges, then we say the series $\sum_{k=1}^{\infty} a_k$ **diverges**, or is **divergent**.

Example 1. The series

$$0.3 + 0.03 + \cdots + 0.00\cdots03 \text{ (k-1 zeros)} + \cdots$$

has the following sequence of partial sums

$$0.3, 0.33, \cdots, 0.33\cdots33 \text{ (k threes)}, \cdots$$

which converges to 1/3. Hence,

$$\sum_{k=1}^{\infty} \frac{3}{10^k} = \frac{1}{3}.$$

Example 2. The series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)} + \cdots,$$

called a **telescoping** series, has partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}.$$

Observe that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \qquad (k = 1, 2, \cdots)$$

yields

$$\begin{aligned} S_n &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) \\ &+ \left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right). \\ &= 1-\frac{1}{n+1}. \end{aligned}$$

Since

$$\lim S_n = \lim\left(1-\frac{1}{n+1}\right) = 1,$$

the series converges to 1, or,

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1.$$

2. Geometric series.

These are series of form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{k-1} + \cdots,$$

where r is called the **common ratio** of the series.

The $n-$the partial sum of this series is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1}.$$

To find a compact form for S_n, consider

$$rS_n = \sum_{k=1}^{n} = ar + ar^2 + \cdots + ar^n.$$

Then

$$(1-r)S_n = S_n - rS_n = a(1-r^n),$$

or

$$S_n = a\frac{1-r^n}{1-r} \qquad (r \neq 1).$$

It follows that

$$\lim S_n = \begin{cases} a/(1-r) & \text{for } |r| < 1 \\ \text{fails to exist} & \text{for } |r| \geq 1 \end{cases}$$

We conclude that the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$

$$\text{converges for } |r| < 1$$

and

$$\text{diverges for } |r| \geq 1 \text{ and } a \neq 0.$$

In the case of convergence, we write

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \qquad \text{for } |r| < 1.$$

Example 3. (i)

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1-1/2} = 2.$$

(ii)

$$\sum_{k=1}^{\infty} \frac{1}{10^k} = \frac{1/10}{1-1/10} = \frac{1}{9}.$$

(iii)

$$\sum_{k=1}^{\infty} \frac{2}{3^{k-1}} = \frac{2}{1-1/3} = 3.$$

(iv) The series

$$\sum_{k=0}^{\infty} \left(\frac{5}{4}\right)^k$$

is divergent since it is a geometric series whose common ratio is $5/4 > 1$.

3. $n-$term test for divergence.

Theorem 1. If the series $\sum_{k=0}^{\infty} a_k$ converges, then $\lim a_n = 0$.

Proof. If S_n is the $n-$th partial sum and $\sum_{k=0}^{\infty} a_k = S$, then

$$a_n = S_n - S_{n-1},$$

and

$$\lim a_n = \lim S_n - \lim S_{n-1} = S - S = 0.$$

Corollary 1. If

(i) $\lim a_n$ fails to exist, or

(ii) $\lim a_n \neq 0$,

then the series $\sum_{k=0}^{\infty} a_k$ diverges.

Example 4. (i) The series

$$\sum_{k=0}^{\infty} \frac{k}{k+1}$$

diverges since $\lim k/(k+1) = 1 \neq 0$.

(ii) The series

$$\sum_{k=1}^{\infty} \frac{e^k}{k}$$

diverges since $\lim e^k/k = \infty \neq 0$.

(iii) The series

$$\sum_{k=1}^{\infty} \sqrt[k]{k}$$

diverges since $\lim \sqrt[k]{k} = 1 \neq 0$.

(iv) The series

$$\sum_{k=1}^{\infty} \frac{k!}{10^k}$$

diverges since $\lim \frac{k!}{10^k} = \infty \neq 0$.

(v) The series

$$\sum_{k=1}^{\infty} \frac{(2k)!}{2^k(k!)}$$

diverges since $\lim \frac{(2k)!}{2^k(k!)} = \infty \neq 0$.

(vi) The series $\sum_{k=1}^{\infty} k \ln(1+1/k)$ diverges since $\lim k \ln(1 + 1/k) = 1 \neq 0$.

Remark 1. The convergence condition $\lim a_n = 0$ for the series $\sum_{k=1}^{\infty} a_k$ is necessary but not sufficient. We show that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

called the **harmonic series**, is divergent even though $\lim 1/n = 0$. To see this, we have:

$$\begin{aligned}
S_{2^1} = S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_{2^2} = S_4 &= S_2 + \frac{1}{3} + \frac{1}{4} > S_2 + \frac{1}{4} + \frac{1}{4} = \frac{4}{2} \\
S_{2^3} = S_8 &= S_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> S_4 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{5}{2} \\
&\vdots \\
S_{2^n} &\geq \frac{n+2}{2} \text{ for every positive integer n.}
\end{aligned}$$

This implies that the sequence $\{S_{2^n}\}$ is divergent, and so is the sequence $\{S_n\}$.

4. Algebra of convergent series.

Theorem 2. Suppose $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Then

(a) **Sum Rule**: $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

(b) **Difference Rule**: $\sum_{k=1}^{\infty}(a_k - b_k) = A - B$.

(c) **Constant Multiple Rule**: $\sum_{k=1}^{\infty} ca_k = cA$.

Example 5. (i)

$$\sum_{k=0}^{\infty}\left(\frac{5}{2^k} - \frac{1}{3^k}\right) = \frac{5}{(1-1/2)} - \frac{1}{(1-1/3)} = \frac{17}{2}.$$

(ii)

$$\sum_{k=1}^{\infty}\left[\frac{1}{k(k+1)} + \frac{1}{2^k}\right] = 1 + \frac{1/2}{1-1/2} = 2.$$

Theorem 3. (a) If series $\sum_{k=1}^{\infty} a_k$ diverges and $c \neq 0$ is a constant, then $\sum_{k=1}^{\infty} ca_k$ diverges.

(b) If series $\sum_{k=1}^{\infty} a_k$ converges and series $\sum_{k=1}^{\infty} b_k$ diverges, then the series $\sum_{k=1}^{\infty}(a_k \pm b_k)$ diverges.

Remark 2. If both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverge, then series $\sum_{k=1}^{\infty}(a_k \pm b_k)$ may converge or diverge.

Example 6. The series $\sum_{k=0}^{\infty}(-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges since its sequence of partial sums $1, 0, 1, 0, 1, 0, \cdots$ diverges. The series $\sum_{k=0}^{\infty}(1/\sqrt{\pi})^k$ converges since it is a geometric series of common ratio $1/\sqrt{\pi} < 1$. Hence the series

$$\sum_{k=0}^{\infty}\left[(-1)^k \pm \left(\frac{1}{\sqrt{\pi}}\right)^k\right]$$

diverges.

5. General remarks.

(a) **Re-indexing of series**: A series can be re-indexed without any alteration as follows:

$$\sum_{k=m}^{\infty} a_k = \sum_{k=m+r}^{\infty} a_{k-r} = \sum_{k=m-r}^{\infty} a_{k+r}$$

for any integer r.

(b) **Addition and deletion of terms**: Adding or deleting a finite number of terms in a series does not change the nature of its convergence. That is why one need not specify the lower index of a series when comes to to decide on its convergence. nonetheless, in case of convergence adding or deleting a finite number of terms in a series may change the sum of the series.

Example 7. We can write

$$\sum_{k=0}^{\infty} \frac{\cos k\pi}{5^k} = \sum_{k=3}^{\infty} \frac{\cos(k-3)\pi}{5^{k-3}} = \sum_{k=-1}^{\infty} \frac{\cos(k+1)\pi}{5^{k+1}}.$$