CHAPTER 11: Infinite Series

§11.6 Alternating Series and Absolute and Conditional Convergence

Section Contents:

(1) Alternating series.

(2) Absolute and conditional convergence.

(3) Rearranging of series.

(4) Ratio and root tests for absolute convergence.

(5) Summary of convergence and divergence tests.

1. Alternating series.

The following are alternating series:

$$\sum_{k=1}^{\infty}(-1)^{k+1} = 1-1+1-1+\cdots+(-1)^{k+1}+\cdots,$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{k+1}}{k}+\cdots,$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2^{k-1}} = 1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots+\frac{1}{2^{k-1}}+\cdots.$$

Definition 1. (**Alternating Series**)

An **alternating series** is of form

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^{k-1}a_k &= a_1 - a_2 + a_3 - a_4 + \cdots \\ &+ (-1)^{k-1}a_k + \cdots,\end{aligned}$$

where each $a_k > 0$.

Theorem 1. (Alternating Series Test; Leibniz)

The alternating series

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^{k-1}a_k &= a_1 - a_2 + a_3 - a_4 + \cdots \\ &+ (-1)^{k-1}a_k + \cdots,\end{aligned}$$

converges if the following conditions hold:

(a) $a_k \geq a_{k+1}$ for all k.

(b) $\lim a_k = 0$.

Sketch of proof. With S_k the $k-$th partial sum we have:

$S_{2k} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2k-1} - a_{2k})$.

By (a), $(a_1 - a_2), (a_3 - a_4), \cdots, (a_{2k-1} - a_{2k}) \geq 0$. Hence,

$$0 \leq S_2 \leq S_4 \leq \cdots \leq S_{2k} \leq \cdots.$$

and the sequence of partial sums $\{S_n\}$ is non-decreasing.

Also, we have:

$$\begin{aligned} S_{2k} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots \\ &- (a_{2k-2} - a_{2k-1}) - a_{2k} \leq a_1. \end{aligned}$$

and the sequence of partial sums $\{S_{2k}\}$ is bounded above.

It follows that the sequence of partial sums $\{S_{2k}\}$ is convergent; namely

$$\lim S_{2k} = S.$$

On the other hand,

$$S_{2k+1} = S_{2k} + a_{2k+1}.$$

Hence, by (b),

$$\lim S_{2k+1} = \lim S_{2k} + \lim a_{2k+1} = S + 0 = S.$$

Thus the sequence of partial sums $\{S_k\}$ is convergent and the alternating series converges.

Example 1. (i) The **alternating harmonic series**

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

converges since (a) $1/k > 1/(k+1)$ for all $k = 1, 2, \cdots$, and (b) $\lim 1/k = 0$.

(ii) The alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{1/3}}$$

converges since (a)$1/k^{1/3} > 1/(k+1)^{1/3}$ for all $k = 1, 2, \cdots$, and (b) $\lim 1/k^{1/3} = 0$.

(iii) The alternating series

$$\sum_{k=1}^{\infty} (-1)^k (1 + 1/k)^k$$

diverges since $\lim(1 + 1/k)^k = e \neq 0$.

(iv) The alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln k}{k}$$

converges since (a) the sequence $\{(\ln k)/k\}$ is decreasing and (b) $\lim \ln k/k = 0$.

Definition 2. (Truncation Error) Let $\sum_{k=1}^{\infty} a_k$ be a convergent series with

$$S = \sum_{k=1}^{\infty} a_k \text{ and } S_n = \sum_{k=1}^{n} a_k.$$

Then the $n-$th **truncation error** of the series is the value $T_n = S - S_n$.

Theorem 2. (Alternating Series Estimation Theorem)

If the alternating series $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ satisfies conditions (a) and (b) of Leibniz theorem, then

$$|T_n| < a_{n+1} \text{ and } \operatorname{sgn} T_n = \operatorname{sgn}(-1)^n.$$

Example 2. (i) The truncation error after 50 terms (T_{50}) of the series alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \cdots - \frac{1}{50} + \frac{1}{51} + \cdots$$

is positive and is less than 1/51; namely $0 < T_{50} < 1/51$.

(ii) The truncation error after 101 terms (T_{101}) of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{k-1}} = 1 - \frac{1}{2} + \cdots + \frac{1}{2^{100}} - \frac{1}{2^{101}} + \cdots$$

is negative and has size less than $1/2^{101}$; namely $-1/2^{101} < T_{101} < 0$.

2. Absolute and conditional convergence.

Definition 3. (Absolute and Conditional Convergence)

A series $\sum_{k=1}^{\infty} a_k$ is said to converge

(a) **absolutely** if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

(b) **conditionally** if it converges but not absolutely; namely $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} |a_k|$ diverges.

Example 3. (i) The alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k+1}/k$ converges conditionally since it converges by Leibniz theorem and the harmonic series diverges.

(ii) The series $\sum_{k=2}^{\infty}(-1)^{k+1}/\ln k$ converges conditionally since, by Leibniz theorem, it converges and, by the comparison test, the series

$\sum_{k=1}^{\infty} 1/\ln k$ diverges. For the latter, observe that $1/\ln k > 1/k$ for all $k > 1$.

(iii) The series $\sum_{k=1}^{\infty} (\sin k)/k^{3/2}$ converges absolutely, by the comparison test, since

$$\left|\frac{\sin k}{k^{3/2}}\right| \leq \frac{1}{k^{3/2}} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \text{ converges.}$$

(iv) The series

$$1 + 1/2 - (1/2)^2 - (1/2)^3 + (1/2)^4$$

$$+(1/2)^5 - (1/2)^6 - (1/2)^7 + \cdots$$

converges absolutely since

$$|\pm(1/2)^k| = (1/2)^k \text{ and } \sum_{k=1}^{\infty} (1/2)^k \text{ converges.}$$

Theorem 3. (Absolute Convergence Theorem)

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges.}$$

Sketch of Proof.

(i) $0 \leq a_k + |a_k| \leq 2|a_k|$.

(ii) $\sum_{k=1}^{\infty} |a_k|$ converges
$\Rightarrow \sum_{k=1}^{\infty}(a_k + |a_k|)$ converges by (i).

(iii) $\sum_{k=1}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty}(a_k + |a_k|)$ converge
$\Rightarrow \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}[(a_k+|a_k|)-|a_k|]$ converges.

Remark 1. The converse of the absolute convergence theorem need not hold: The alternating harmonic series converges but the harmonic series diverges.

Example 4. (i) The series $\sum_{k=1}^{\infty}(\sin k)/k^{3/2}$ converges since it converges absolutely.

(ii) The series $\sum_{k=1}^{\infty}(-1)^{\lfloor k/3 \rfloor}/k^3$ converges since it converges absolutely.

(iii) The series

$$1 + 1/2 - (1/2)^2 - (1/2)^3 + (1/2)^4$$

$$+(1/2)^5 - (1/2)^6 - (1/2)^7 + \cdots$$

converges since it converges absolutely.

3. Rearranging of series.

Definition 4. A series $\sum_{k=1}^{\infty} b_k$ is called a **rearrangement** of a series $\sum_{k=1}^{\infty} a_k$ if the sequence $\{b_k\}$ is a rearrangement of the sequence $\{a_k\}$.

Theorem 4. (The Rearrangement Theorem For Absolutely Convergent Series)

Suppose

(a) $\sum_{k=1}^{\infty} b_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$.

(b) $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Then $\sum_{k=1}^{\infty} b_k$ converges absolutely and

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k.$$

Example 5. The series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

converges absolutely. By rearranging its terms indefinitely we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} &= \left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots + \frac{1}{(2k-1)^2} + \cdots\right] \\
&- \left[\frac{1}{2^2} + \frac{1}{4^2} + \cdot + \frac{1}{(2k)^2} + \cdots\right] \\
&= \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2}.
\end{aligned}$$

Ratio and root tests for absolute convergence.

Theorem 5.(**The ratio test for absolute convergence**)

Suppose the series $\sum_1^\infty a_k$ satisfies $a_k \neq 0$ for all $k \geq 1$, and suppose

$$\lim \left| \frac{a_{k+1}}{a_k} \right| = L \text{ (possibly } \infty.)$$

Then

(a) $L < 1 \Rightarrow \sum_1^\infty a_k$ converges (absolutely).

(b) $L > 1 \Rightarrow \sum_1^\infty a_k$ diverges.

(c) $L = 1 \Rightarrow \sum_1^\infty a_k$ is undecided.

Example 6. (i) The series $\sum_1^\infty (-1)^k/k!$ converges since

$$\lim \left| \frac{(-1)^k/(k+1)!}{1/k!} \right| = \lim \frac{1}{k+1} = 0 < 1.$$

(ii) The series $\sum_1^\infty (-1)^k k^3/10^k$ converges since

$$\lim \left| (-1)^k \frac{(k+1)^3/10^{k+1}}{k^3/10^k} \right| =$$

$$\frac{1}{10} \left(\lim \frac{k+1}{k} \right)^3 = \frac{1}{10} < 1.$$

(iii) The series $\sum_1^\infty (-1)^k (2k)!/(k!)^2$ diverges since

$$\left| (-1)^k \frac{(2k+2)!/[(k+1)!]^2}{(2k)!/[(k)!]^2} \right| = \lim \frac{4k+2}{k+1} = 4 > 1.$$

(iv) The series $\sum_1^\infty (-1)^k k^k/k!$ diverges since

$$\lim \left| (-1)^k \frac{(k+1)^{k+1}/(k+1)!}{k^k/k!} \right| =$$

$$\lim \left(\frac{k+1}{k} \right)^k = e > 1.$$

Theorem 6. (**The root test for absolute convergence**)

Suppose

$$\lim \sqrt[k]{|a_k|} = L \text{ (possibly } \infty.)$$

Then

(a) $L < 1 \Rightarrow \sum_1^\infty a_k$ converges absolutely.

(b) $L > 1 \Rightarrow \sum_1^\infty a_k$ diverges.

(c) $L = 1 \Rightarrow \sum_1^\infty a_k$ is undecided.

Example 7. (i) The series $\sum_2^\infty (-1)^k/(\ln k)^k$ converges since

$$\lim \sqrt[k]{1/(\ln k)^k} = \lim \frac{1}{\ln k} = 0 < 1.$$

(ii) The series $\sum_1^\infty (-1)^k 2^k/k^3$ diverges since

$$\lim \sqrt[k]{2^k/k^3} = 2 \lim \left(\frac{1}{\sqrt[k]{k}}\right)^3 = 2 > 1.$$

5. Summary of convergence and divergence tests. See Textbook: Figure 8.2