CHAPTER 11: Infinite Series

§11.5 The Ratio and Root Tests Tests

Section Contents:

(1) Ratio Test.

(ii) Root Test.

1. Ratio test

Theorem 1. (**The ratio test**)

Suppose the series $\sum_1^\infty a_k$ satisfies $a_k > 0$ for all $k \geq 1$, and suppose

$$\lim \frac{a_{k+1}}{a_k} = L \text{ (possibly } \infty.)$$

Then

(a) $L < 1 \Rightarrow \sum_1^\infty a_k$ converges.

(b) $L > 1 \Rightarrow \sum_1^\infty a_k$ diverges.

(c) $L = 1 \Rightarrow \sum_1^\infty a_k$ is undecided.

Example 1. (i) The series $\sum_1^\infty 1/k!$ converges since

$$\lim \frac{1/(k+1)!}{1/k!} = \lim \frac{1}{k+1} = 0 < 1.$$

(ii) The series $\sum_1^\infty k^3/10^k$ converges since

$$\lim \frac{(k+1)^3/10^{k+1}}{k^3/10^k} = \frac{1}{10}\left(\lim \frac{k+1}{k}\right)^3 = \frac{1}{10} < 1.$$

(iii) The series $\sum_1^\infty (2k)!/(k!)^2$ diverges since

$$\frac{(2k+2)!/[(k+1)!]^2}{(2k)!/[(k)!]^2} = \lim \frac{4k+2}{k+1} = 4 > 1.$$

(iv) The series $\sum_1^\infty k^k/k!$ diverges since

$$\lim \frac{(k+1)^{k+1}/(k+1)!}{k^k/k!} = \lim \left(\frac{k+1}{k}\right)^k = e > 1.$$

1. Root test

Theorem 2. (The root test)

Suppose the series $\sum_1^\infty a_k$ satisfies $a_k \geq 0$ for all $k \geq 1$, and suppose

$$\lim \sqrt[k]{a_k} = L \text{ (possibly } \infty.)$$

Then

(a) $L < 1 \Rightarrow \sum_1^\infty a_k$ converges.

(b) $L > 1 \Rightarrow \sum_1^\infty a_k$ diverges.

(c) $L = 1 \Rightarrow \sum_1^\infty a_k$ is undecided.

Example 2. (i) The series $\sum_2^\infty 1/(\ln k)^k$ converges since

$$\lim \sqrt[k]{1/(\ln k)^k} = \lim \frac{1}{\ln k} = 0 < 1.$$

(ii) The series $\sum_1^\infty 2^k/k^3$ diverges since

$$\lim \sqrt[k]{2^k/k^3} = 2 \lim \left(\frac{1}{\sqrt[k]{k}}\right)^3 = 2 > 1.$$

(iii) The root test fails with the series $\sum_1^\infty (1 - 1/k)^k$ since

$$\lim \sqrt[k]{(1 - 1/k)^k} = \lim(1 - 1/k) = 1.$$

However, the series diverges by the $k-$th term test since

$$\lim(1 - 1/k)^k = 1/e \neq 0.$$